ACCESS TO ENGINEERING - MATHEMATICS 2

ADEDEX428

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4. Integral Calculus

4.1. **Introduction to Integration.**

As was the case with the chapter on differential calculus, for most of this chapter we will concentrate on the mechanics of how to integrate functions. However we will first give an indication as to what we are actually doing when we integrate functions. This can be made rigorous mathematically but in this course we just want to get an intuitive idea of what is going on.

Suppose we want to find the area lying between the graph of a function and the x-axis between two given points a and b. Then one way of doing this would be to approximate this area by the area of rectangles which lie under the graph, as shown in Figure 1. The reason we use rectangles is because it is easy to calculate their area, it is simply their height times their width.

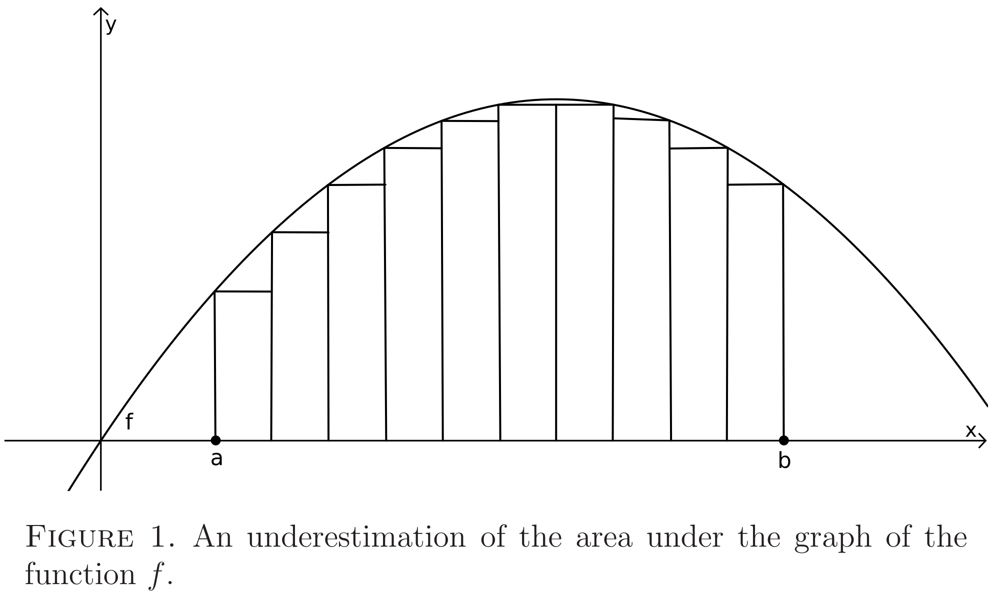

Figure 1. An underestimation of the area under the graph of the function f.

Of course the problem with this approach is that we will usually underestimate the area under the curve since we are not including the area above the rectangles and under the graph. One possible solution would be to make the width of the rectangles smaller and smaller. In this way we would hopefully get a better approximation to the area under the curve. However we can not be sure that this would be the case if we are dealing with a really strange function.

Another approach is to overestimate the area by putting the rectangles above the curve as Shown in Figure 2.

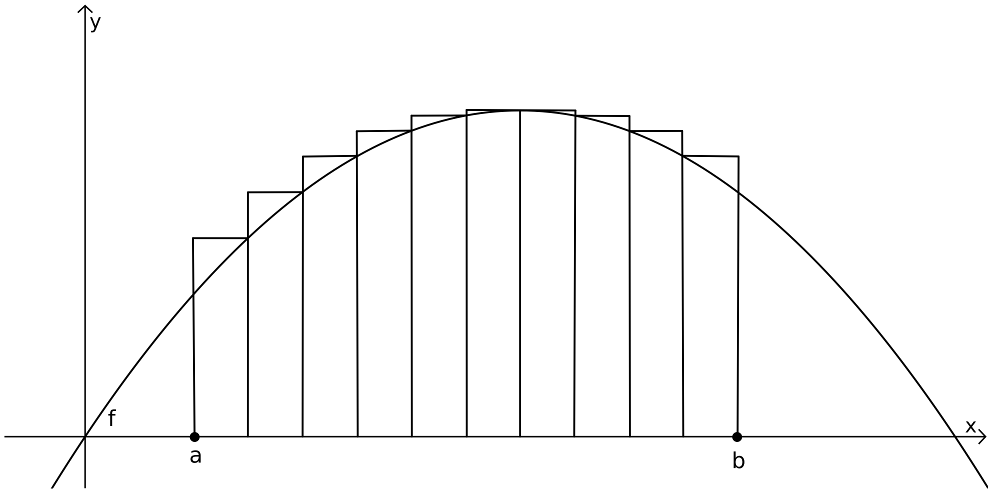

FIGURE 2. An overestimation of the area under the graph of the function f.

You might point out that this doesn't get us any further and you would be correct. Clearly it is no better to have an overestimation of the area. However the clever bit is that we can try and reduce the overestimation by changing the widths of the rectangles and we can try and reduce the underestimation the same way (using different rectangles). If we can get both the overestimation and the underestimation of the area to be 'close' to a given number A then we say that the function f is integrable on the interval $[a, b]$ and we write $\int_a^b f(x)\, dx = A$. In this case the area under the curve *is* A. The number $\int_a^b f(x)\, dx$ has a special name.

Definition 4.1.1 (Indefinite Integral)**.** If a function f is integrable on the interval $[a, b]$, then the number $\int_a^b f(x)\, dx$ is called the *indefinite integral* of f from a to b. The function f is called the *integrand.*

In Figures 1 and 2, we have given an example of a function that lies above the x-axis between the points a and b but the area is a 'signed area'. That is if part of the graph of f lies below the x-axis then this area is counted as negative. For example in Figure 3, the integral $\int_a^b f(x)\, dx$ represents the area in red minus the area in green. This means that if we are going to use integrals to calculate areas rather than signed areas, we have to first find which parts of the graph lie above the x-axis and which parts lie below. In the case of Figure 3, the actual area that lies between the graph of f and the x-axis between the points a and b (i.e., the area of the red portion plus the area of the green portion) is $\displaystyle\int_a^c f(x)\,dx - \int_c^b f(x)\,dx$. Note that we have to put a minus sign before the integral $\displaystyle\int_c^b f(x)\,dx$ to allow for the fact that $\displaystyle\int_c^b f(x)\,dx$ gives minus the green area.

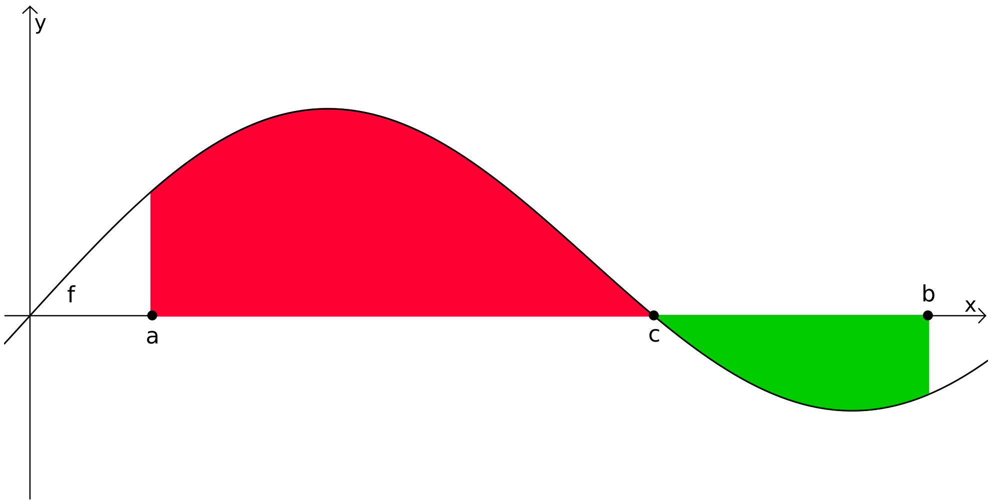

FIGURE 3. Signed area under the graph of the function f.

4.2. **The Fundamental Theorem of Calculus.**

It is all very well defining an integral as we did in Section 4.1 but in practice it is almost impossible to use this definition to actually calculate areas. Luckily, the Fundamental Theorem of Calculus comes to our rescue. There are several slightly different forms of this theorem that you may meet in your studies but the one we are going to use is the following.

Theorem 4.2.1 (The Fundamental Theorem of Calculus)**.** *Let F and f be functions defined on an interval $[a, b]$ such that f is continuous and such that the derivative of F is f. Then*

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

Remark 4.2.2. Although this result is taught quite early on in your mathematical career, it is a most remarkable and very deep result. It connects two seemingly completely unrelated concepts. Firstly there is the derivative of a function, which gives the slope of a tangent to a curve and then there is the integral of a function, which calculates the area under the curve.

The function F that appears in Theorem 4.2.1 has a special name.

Definition 4.2.3 (Antiderivative)**.** Let F be any function such that the derivative of F is equal to another function f. Then F is said to be an *antiderivative* of f.

Note that the antiderivative of a function is not unique. If F is any antiderivative of f and if c is a constant, then it follows from the sum rule and the fact that the derivative of a constant is zero, that $F+c$ is also an antiderivative of f. However, when using The Fundamental Theorem of Calculus, it doesn't matter if we use F or $F+c$ since $(F+c)(b)-(F+c)(a) = F(b)+c-(F(a)+c) = F(b)-F(a)$. That is the constant will always cancel out.

The function $F+c$, where c is a arbitrary constant, also has a special name.

Definition 4.2.4 (Indefinite integral)**.** Let F be any function such that the derivative of F is equal to another function f and let c be an arbitrary constant. Then $F+c$ is said to be an *indefinite integral* of f and the c is said to be a *constant of integration*. This is written as $\displaystyle\int f(x)\,dx = F(x)+c$. That is, there is no a or b on the integral sign.

Although we have a lot of progress theoretically, we have still not actually calculated any integrals and that is what we will turn our attention to next.

4.3. **Some Common Integrals.**

As with differentiation, we start with some basic integrals and then use these to integrate a wide range of functions using various rules and techniques. The basic integrals that you will need in this course are collected together in Table 1. The main thing is to learn how to use them rather than learning them off by heart, since this table will be included in the exam paper. Note that in the table, c will stand for an arbitrary constant.

$f(x)$	$\int f(x)\,dx$	Comments
k	$kx+c$	Here k is any real number
x^n	$\frac{1}{n+1}x^{n+1}+c$	Here we must have $n \neq -1$
$\frac{1}{x}$	$\ln(x)+c$	Here we must have $x > 0$
e^{ax}	$\frac{1}{a}e^{ax}+c$	
$\sin(ax)$	$-\frac{1}{a}\cos(ax)+c$	Note the change of sign
$\cos(ax)$	$\frac{1}{a}\sin(ax)+c$	

TABLE 1. Some common integrals

Warning 4.3.1. (1) As was the case with derivatives, the integrals of $\sin(ax)$ and $\cos(ax)$ are only valid if x is in radians. If x is in degrees then extra constants are needed.
(2) Note that the minus sign occurs with the integral of $\sin(ax)$, rather than the integral of $\cos(ax)$, where it appeared when we were differentiating.

As always, some examples will make things clearer. First of all we will give some indefinite integrals in Table 2.

Remark 4.3.2. If you want to check your answer when you have found a definite integral then all you need to do is to differentiate your answer. You should always get back to the function you started with.

In Example 4.3.3 I have given a few examples of definite integrals but really finding the indefinite integral is the hard part. Once you have this, finding the definite integral is just a matter of substituting numbers into the formula. Please do remember however that the value of the antiderivative at the lower limit has to be subtracted from the value of the antiderivative at the upper limit. Also note that when calculating definite integrals, we ignore the constant of integration c since it always cancels out.

Example 4.3.3. (1) Calculate the definite integral $\displaystyle\int_1^2 x^2\,dx$.

$$\int_1^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_1^2 = \frac{1}{3}2^3 - \frac{1}{3}1^3 = \frac{7}{3}.$$

(2) Calculate the definite integral $\displaystyle\int_0^\pi \sin(2x)\,dx$.

$$\begin{aligned}\int_0^\pi \sin(2x)\,dx &= \left[-\frac{1}{2}\cos(2x)\right]_0^\pi \\ &= -\frac{1}{2}\cos(2\pi) - \left(-\frac{1}{2}\cos(0)\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0.\end{aligned}$$

Note that in this case the integral is zero since the area above the x-axis cancels out the area below the x-axis.

(3) Calculate the definite integral $\displaystyle\int_{-2}^{-1} e^{-4x}\,dx$.

$$\int_{-2}^{-1} e^{-4x}\,dx = \left[-\frac{1}{4}e^{-4x}\right]_{-2}^{-1} = -\frac{1}{4}e^4 - \left(-\frac{1}{4}e^8\right) = \frac{e^8 - e^4}{4}.$$

As expected this integral is positive since $e^x > 0$ for all values of x (i.e., the graph of $f(x) = e^x$ lies above the x-axis).

$f(x)$	$\int f(x)\,dx$	Comments
0	c	
2	$2x + c$	
-4	$-4x + c$	
$-\pi$	$-\pi x + c$	$-\pi$ is just a number
e	$ex + c$	e is just a number
$\cos(1)$	$\cos(1)x + c$	$\cos(1)$ is just a number
x	$\frac{1}{2}x^2 + c$	Since $x = x^1$, $n = 1$
x^3	$\frac{1}{4}x^4 + c$	Here we take $n = 3$
x^{-4}	$-\frac{1}{3}x^{-3} + c = -\frac{1}{3x^3} + c$	Here we take $n = -4$
x^{π}	$\frac{1}{\pi+1}x^{\pi+1} + c$	π is just a number
x^{-e}	$\frac{1}{-e+1}x^{-e+1} + c$	e is just a number
e^x	$e^x + c$	Here we take $a = 1$
e^{5x}	$\frac{1}{5}e^{5x} + c$	Here we take $a = 5$
e^{-7x}	$-\frac{1}{7}e^{-7x} + c$	Here we take $a = -7$
e^{ex}	$\frac{1}{e} \cdot e^{ex} + c = e^{ex-1} + c$	Here we take $a = e$
$\sin(x)$	$-\cos(x) + c$	Here we take $a = 1$
$\sin(3x)$	$-\frac{1}{3}\cos(3x) + c$	Here we take $a = 3$
$\sin(-2x)$	$\frac{1}{2}\cos(-2x) + c$	Here we take $a = -2$
$\sin(-\pi x)$	$\frac{1}{\pi}\cos(-\pi x) + c$	Here we take $a = -\pi$
$\cos(x)$	$\sin(x) + c$	Here we take $a = 1$
$\cos(4x)$	$\frac{1}{4}\sin(4x) + c$	Here we take $a = 4$
$\cos(-5x)$	$-\frac{1}{5}\sin(-5x) + c$	Here we take $a = -5$
$\cos(\pi x)$	$\frac{1}{\pi}\sin(\pi x) + c$	Here we take $a = \pi$

TABLE 2. Some examples of indefinite integrals

4.4. **The Sum and Multiple Rules.**

As was the case with differentiation, although the integrals in Table 1 are very useful, we would not get very far if these were the only functions we could integrate. Luckily there are rules that allow us to integrate more complicated functions. The first two of these are almost identical to the equivalent ones for differentiation.

Theorem 4.4.1 (The Sum Rule for Integration)**.** *Let $f\colon (a,b) \to \mathbb{R}$ and $g\colon (a,b) \to \mathbb{R}$, then the definite integral of $f+g$ on the interval $[a,b]$ is given by*

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx,$$

provided the integrals of f and g exist.

All this says is that if we want to integrate a sum of two functions then all we have to do is integrate them separately and add the integrals.

Remark 4.4.2. As you might expect there is an equivalent rule for indefinite integrals:

$$\int (f+g)(x)\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

Note that when you have a sum like this you only need to include one constant of integration. This is since if you add an arbitrary constant to an arbitrary constant you just get an arbitrary constant.

Here are a couple of examples of the use of the Sum Rule.

Example 4.4.3. (1) Evaluate the definite integral $\displaystyle\int_{-1}^{1} x^4 + e^{-x}\,dx$.

$$\begin{aligned}\int_{-1}^{1} x^4 + e^{-x}\,dx &= \int_{-1}^{1} x^4\,dx + \int_{-1}^{1} e^{-x}\,dx\\ &= \left[\frac{1}{5}x^5\right]_{-1}^{1} + \left[-e^{-x}\right]_{-1}^{1}\\ &= \frac{1}{5}1^5 - \frac{1}{5}(-1)^5 + (-e^{-1}) - (-e^{1})\\ &= \frac{2}{5} + e - e^{-1}.\end{aligned}$$

(2) Find the indefinite integral $\displaystyle\int \frac{1}{x} + \cos(-3x)\,dx$.

Provided $x > 0$ (so that $\displaystyle\int \frac{1}{x}\,dx = \ln(x) + c$),

$$\begin{aligned}\int \frac{1}{x} + \cos(-3x)\,dx &= \int \frac{1}{x}\,dx + \int \cos(-3x)\,dx\\ &= \ln(x) - \frac{1}{3}\sin(-3x) + c.\end{aligned}$$

As was the case with differentiation, the second rule that will enable us to integrate a larger range of functions is the Multiple Rule.

Theorem 4.4.4 (The Multiple Rule for Integration). *Let $f\colon (a,b) \to \mathbb{R}$ and let $k \in \mathbb{R}$ (here I will use k instead of c to avoid confusion with the constant of integration c). Then the definite integral of kf over the interval $[a,b]$ is given by*

$$\int_a^b (kf)(x)\,dx = k\int_a^b f(x)\,dx,$$

provided the integral of f exists.

All this says is that if we want to integrate a constant multiple of a function, then all we have to do is first integrate the function and then multiply by the constant.

Remark 4.4.5. Of course, there is a corresponding Multiple Rule for indefinite integrals:

$$\int (kf)(x)\,dx = k\int f(x)\,dx.$$

Here are a couple of examples of how the Multiple Rule works.

Example 4.4.6. (1) Evaluate the definite integral $\displaystyle\int_1^2 -\frac{1}{2x}\,dx$.

$$\begin{aligned}\int_1^2 -\frac{1}{2x}\,dx &= -\frac{1}{2}\int_1^2 \frac{1}{x}\,dx \\ &= -\frac{1}{2}\left[\ln(x)\right]_1^2 \\ &= -\frac{1}{2}\left(\ln(2) - \ln(1)\right) \\ &= -\frac{\ln(2)}{2}.\end{aligned}$$

Note that since the graph of $f(x) = -\dfrac{1}{2x}$ lies below the x-axis on the interval $[1,2]$, the integral $\displaystyle\int_1^2 -\frac{1}{2x}\,dx$ must be negative.

(2) Find the indefinite integral $\displaystyle\int 3e^{4x}\,dx$.

$$\int 3e^{4x}\,dx = 3\int e^{4x}\,dx = 3\frac{1}{4}e^{4x} + c = \frac{3e^{4x}}{4} + c.$$

Here we just write c rather than $3c$ since three times an arbitrary constant is still just an arbitrary constant.

As you would expect, both the sum and multiple rules can be used at the same time. Here are a couple of examples of this.

Example 4.4.7. (1) Evaluate the definite integral $\int_{-\pi}^{\pi} 2\sin(3x) - 4e^x \, dx$.

$$\begin{aligned}
\int_{-\pi}^{\pi} 2\sin(3x) - 4e^x \, dx &= \int_{-\pi}^{\pi} 2\sin(3x) \, dx + \int_{-\pi}^{\pi} -4e^x \, dx \\
&= 2\int_{-\pi}^{\pi} \sin(3x) - 4\int_{-\pi}^{\pi} e^x \, dx \\
&= 2\left[-\frac{1}{3}\cos(3x)\right]_{-\pi}^{\pi} - 4\left[e^x\right]_{-\pi}^{\pi} \\
&= 2\left[-\frac{1}{3}\cos(3\pi) - \left(-\frac{1}{3}\cos(-3\pi)\right)\right] - 4\left[e^{\pi} - e^{-\pi}\right] \\
&= 2\left[\frac{1}{3} - \frac{1}{3}\right] - 4\left[e^{\pi} - e^{-\pi}\right] \\
&= 4\left(e^{-\pi} - e^{\pi}\right).
\end{aligned}$$

(2) Find the indefinite integral $\int -\frac{1}{6x} + 5x^5 \, dx$.

Provided $x > 0$ (so that $\int \frac{1}{x} \, dx = \ln(x) + c$),

$$\begin{aligned}
\int -\frac{1}{6x} + 5x^5 \, dx &= \int -\frac{1}{6x} \, dx + \int 5x^5 \, dx \\
&= -\frac{1}{6}\int \frac{1}{x} \, dx + 5\int x^5 \, dx \\
&= -\frac{1}{6}\ln(x) + 5\left(\frac{1}{6}x^6\right) + c \\
&= \frac{5x^6 - \ln(x)}{6} + c.
\end{aligned}$$

Again note we only have the one arbitrary constant.

4.5. **Integration by Substitution.**

Unfortunately the sum and multiple rules are the only rules that carry over directly from differentiation to integration. While there are rules for integration, they are not quite as direct as the rules for differentiation and it can often be harder to decide which rule to use. Because of this it is even more important to do lots of practice problems for integration, since it is only through experience that you will learn which rule is likely to be the best one to use. It is also the case that it is easy to write down functions that can't be integrated algebraically (although there are numerical methods that can be used, we won't be looking at these in this course). For example the function $f(x) = e^{x^2}$ can't be integrated.

The first technique we will look at is integration by substitution. This can be written down as a theorem but I feel that approaching it in this way can make it look more difficult than it actually is. It is far better in my opinion to learn this technique through looking at some examples and this is what we will do now.

Example 4.5.1. Evaluate the definite integral $\int_0^1 (x+3)^{10}\,dx$.

One way of approaching this problem would be to expand $(x+3)^{10}$ and then integrate the resulting expression. However this would take a lot of work and there would be lots of scope for errors. Instead we will use integration by substitution. The key to the technique is to note that $f(x) = (x+3)^{10}$ can also be written as $f(x) = u^{10}$, if we let $u = x+3$.

The three things we have to do now are:

(i) Express the function $f(x) = (x+3)^{10}$ in terms of u. Note that in general in this step, there will be some function of x 'left over'. With luck this 'left over' bit will cancel with the expression we obtain in (ii).
(ii) Express dx in terms of du, u and x.
(iii) Change the limts of integration to be in terms of u rather than x.

If this method is going to work we will now have an integral with respect to u and with no x's appearing anywhere. If there are x's still left then we will either have to try a different substitution or we will have to try a different method altogether.

In this case we do each of these as follows:

(i) Since $u = x+3$, the function $(x+3)^{10}$ becomes u^{10}.
(ii) For this we use the formula $dx = \frac{dx}{du}du = \frac{du}{du/dx}$. In this case $\frac{du}{dx} = 1$, so $dx = \frac{du}{1} = du$.
Note that dx and du are really not variables but in this particular situation, you can treat them as if they were.
(iii) When $x = 0$, $u = 0+3 = 3$ and when $x = 1$, $u = 1+3 = 4$. Thus the lower and upper limits of integration become 3 and 4 respectively.

Putting all this together we obtain

$$\begin{aligned}
\int_0^1 (x+3)^{10}\,dx &= \int_3^4 u^{10}\,du \quad \text{Note there are no } x\text{'s here.} \\
&= \left[\frac{1}{11}u^{11}\right]_3^4 \\
&= \frac{1}{11}4^{11} - \frac{1}{11}3^{11} \\
&= \frac{4^{11} - 3^{11}}{11}.
\end{aligned}$$

Example 4.5.2. Find the indefinite integral $\displaystyle\int \frac{6x^2+4x+2}{x^3+x^2+x+1}\,dx$.

First note that when we are using integration by substitution to find an indefinite integral, then clearly we only need to perform the first two steps above. However, the main question is what substitution will we make? Looking at the integrand, we see that the numerator is a multiple of the derivative of the denominator. In cases like this, when one part of the integrand is a multiple of the derivative of another part, then a good strategy is to let u equal the undifferentiated bit. So we let $u = x^3 + x^2 + x + 1$. Then the steps are:

(i) We have $\displaystyle\frac{6x^2+4x+2}{x^3+x^2+x+1} = \frac{6x^2+4x+2}{u}$. Note that in this case we still have a function of x in the numerator.

(ii) Since $\displaystyle\frac{du}{dx} = 3x^2+2x+1$, we have $\displaystyle dx = \frac{du}{du/dx} = \frac{du}{3x^2+2x+1}$.

Putting this together we have

$$\begin{aligned}
\int \frac{6x^2+4x+2}{x^3+x^2+x+1}\,dx &= \int \frac{6x^2+4x+2}{u}\cdot\frac{du}{3x^2+2x+1}\\
&= \int \frac{2}{u}\,du\\
&= 2\int \frac{1}{u}\,du\\
&= 2\ln(u)+c\\
&= 2\ln\left(x^3+x^2+x+1\right)+c.
\end{aligned}$$

Remark 4.5.3. (1) The step where we have written $\displaystyle\int \frac{6x^2+4x+2}{u}\cdot\frac{du}{3x^2+2x+1}$ is a bit dubious notationally, since the integral is with respect to u but there are also x's in the expression. However I recommend that you include this step, at least in this course.

(2) Once we have a definite integral then it is a good idea to differentiate it and make sure we get the original function. I will leave it to you to differentiate $f(x) = 2\ln(x^3+x^2+x+1)+c$ and make sure you get $\displaystyle f'(x) = \frac{6x^2+4x+2}{x^3+x^2+x+1}$.

Example 4.5.4. Evaluate the definite integral $\displaystyle\int_0^{\frac{\sqrt{\pi}}{2}} x\cos\left(x^2\right)\,dx$.

Again in this example we see that x is a multiple of the derivative of x^2, so let us try the substitution $u = x^2$. The three steps are:

(i) Since $u = x^2$, the function $x\cos(x^2)$ becomes $x\cos(u)$. We have an x left over here. In this case we could write $x = \sqrt{u}$ but this would not be a good idea since we are hoping that the x will cancel with part of the expression we get from (ii).

(ii) Since $\dfrac{du}{dx} = 2x$, $dx = \dfrac{du}{du/dx} = \dfrac{du}{2x}$.

(iii) When $x = 0$, $u = 0^2 = 0$ and when $x = \dfrac{\sqrt{\pi}}{2}$, $u^2 = \left(\dfrac{\sqrt{\pi}}{2}\right)^2 = \dfrac{\pi}{4}$. Thus the lower and upper limits of integration become 0 and $\dfrac{\pi}{4}$ respectively.

Putting this together we obtain

$$\begin{aligned}
\int_0^{\frac{\sqrt{\pi}}{2}} x\cos\left(x^2\right)\,dx &= \int_{u=0}^{u=\frac{\pi}{4}} x\cos(u)\cdot\frac{du}{2x} \\
&= \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos(u)\,du \\
&= \frac{1}{2}[\sin(u)]_0^{\frac{\pi}{4}} \\
&= \frac{1}{2}\left[\sin\left(\frac{\pi}{4}\right) - \sin(0)\right] \\
&= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{2}}{4}.
\end{aligned}$$

Example 4.5.5. Find the indefinite integral $\displaystyle\int 2\cos(x)e^{\sin(x)}\,dx$.

Here we note that $\cos(x)$ is the derivative of $\sin(x)$, so we try the substitution $u = \sin(x)$. The steps are as follows.

(i) We have $2\cos(x)e^{\sin(x)} = 2\cos(x)e^u$. Note that again we have a function of x left over'. It is possible to express $2\cos(x)$ as a function of u but agin this would not be a good idea.

(ii) Since $\dfrac{du}{dx} = \cos(x)$, we have $dx = \dfrac{du}{du/dx} = \dfrac{du}{\cos(x)}$.

Putting this together we obtain

$$\begin{aligned}
\int 2\cos(x)e^{\sin(x)}\,dx &= \int 2\cos(x)e^u\cdot\frac{du}{\cos(x)} \\
&= \int 2e^u\,du \\
&= 2\int e^u\,du \\
&= 2e^u + c \\
&= 2e^{\sin(x)} + c.
\end{aligned}$$

Again you should check that on differentiating $2e^{\sin(x)} + c$ you get $2\cos(x)e^{\sin(x)}$.

4.6. **Integration by Parts.**

The next technique is used to transform integrals of the form $\int f(x)g'(x)\,dx$ into something easier to integrate. In this course $f(x)$ will usually be a power of x (or a polynomial in x) but we will also look at one other interesting case.

Provided all the integrals exist then the integration by parts formula for indefinite integrals says that

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

and the integration by parts formula for definite integrals says that

$$\int_a^b f(x)g'(x)\,dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,dx.$$

Note that in the original integral, we have f times the derivative of g, while in the integral on the right we have g times the derivative of f.

Remark 4.6.1.
(1) There is no constant of integration in the integration by parts formula for definite integrals since we can just include it in the integral $\int f'(x)g(x)\,dx$.
(2) In order to use the method of integration by parts we have to be able to integrate the function that we call $g'(x)$. Sometimes this will give us a clue as to which function we will call f and which function we will call g.
(3) It is possible when using integration by parts to end up with a more difficult integral than we started with. If this happens then it means we will have to go back to the drawing board - either we will have to choose different functions for f and g or maybe integration by parts is not a suitable method for the particular function we are trying to integrate.

Now we will look at some examples to see how the method works in practice.

Example 4.6.2. Find the indefinite integral $\int xe^x\,dx$.

If we look at this integral we note that if we could get rid of the x then we could integrate the e^x, so this suggests that we let $f(x) = x$ and $g'(x) = e^x$.
(Note that if we let $f(x) = e^x$ and $g'(x) = x$, then although we could perform integration by parts, we would end up with a more complicated integral since we would have $g(x) = \frac{1}{2}x^2$ and $f'(x) = e^x$, so we would have $\int f(x)g'(x)\,dx = \int \frac{1}{2}x^2e^x\,dx$).
With $f(x) = x$ and $g'(x) = e^x$, we obtain $f'(x) = 1$ and $g(x) = e^x$, so the integration by parts formula yields

$$\int xe^x\,dx = xe^x - \int e^x\,dx.$$

We can now easily finish the integration to get $\int xe^x \, dx = xe^x - e^x + c$.

Let us now use integration by parts with a definite integral.

Example 4.6.3. Evaluate the definite integral $\int_0^{\frac{\pi}{2}} 2x \sin(3x) \, dx$.

Here we let $f(x) = 2x$ and $g'(x) = \sin(3x)$, so that $f'(x) = 2$ and $g(x) = -\frac{1}{3}\cos(3x)$. Thus, using the integration by parts formula, we obtain

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} 2x \sin(3x) \, dx &= \left[-\frac{2x}{3}\cos(3x)\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\frac{2}{3}\cos(3x) \, dx \\
&= -\frac{\pi}{3}\cos\left(\frac{3\pi}{2}\right) - (-0) + \int_0^{\frac{\pi}{2}} \frac{2}{3}\cos(3x) \, dx \\
&= \int_0^{\frac{\pi}{2}} \frac{2}{3}\cos(3x) \, dx \\
&= \left[\frac{2}{9}\sin(3x)\right]_0^{\frac{\pi}{2}} \\
&= \frac{2}{9}\sin\left(\frac{3\pi}{2}\right) - \frac{2}{9}\sin(0) \\
&= -\frac{2}{9}.
\end{aligned}$$

The next example is a little bit different since it appears at first sight that we can't use integration by parts since we don't have a product of functions.

Example 4.6.4. Find the indefinite integral $\int \ln(x) \, dx$.

The key here is to note that we can write $\ln(x) = 1 \cdot \ln(x)$ and let $f(x) = \ln(x)$ and $g'(x) = 1$, so that $f'(x) = \frac{1}{x}$ and $g(x) = x$. Then, using the integration by parts formula, we obtain

$$\int \ln(x) \, dx = x\ln(x) - \int \frac{1}{x} x \, dx = x\ln(x) - \int 1 \, dx = x\ln(x) - x + c.$$

Remark 4.6.5. Note that differentiating on the log term will also enable us to integrate functions of the form $y = x^n \ln(x)$.

We will finish this section by looking at an example where we have to use the integration by parts formula twice.

Example 4.6.6. Evaluate the definite integral $\int_0^{\pi} x^2 \cos(x) \, dx$.

For the first application of the formula, we let $f(x) = x^2$ and $g'(x) = \cos(x)$, so that

$f'(x) = 2x$ and $g(x) = \sin(x)$. Then, using the integration by parts formula, we obtain

$$\int_0^\pi x^2\cos(x)\,dx = \left[x^2\sin(x)\right]_0^\pi - \int_0^\pi 2x\sin(x)\,dx = 0 - 0 + \int_0^\pi -2x\sin(x)\,dx.$$

For the second application of the formula, we let $f(x) = -2x$ and $g'(x) = \sin(x)$, so that $f'(x) = -2$ and $g(x) = -\cos(x)$. Then, using the integration by parts formula again, we obtain

$$\begin{aligned}\int_0^\pi -2x\sin(x)\,dx &= [2x\cos(x)]_0^\pi - \int_0^\pi 2\cos(x)\,dx \\ &= -2\pi - 0 - [2\sin(x)]_0^\pi \\ &= -2\pi - (0-0) \\ &= -2\pi.\end{aligned}$$

Thus $\displaystyle\int_0^\pi x^2\cos(x)\,dx = -2\pi$.

4.7. **Integration Using Partial Fractions.**

The last technique of integration we will look at is integration using partial fractions. Really this is not so much a technique of integration, it is more a technique used to express algebraic expressions in a different form. It is used in situations where we are dealing with functions of the form $\dfrac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are both polynomials with the degree of f being less than the degree of g and where g can be factorised.

Remark 4.7.1. This technique can be extended to the case where the degree of f is greater than or equal to the degree of g but in this case we first have to divide f by g using polynomial long division. We covered this in the first semester, but I won't give you any questions on integration requiring it since it would make the questions too long.

Now let us look at some examples to see how this technique works in practice. The easiest case is where g is a quadratic and can be factored into two different linear factors

Example 4.7.2. Find the indefinite integral $\displaystyle\int \frac{-4}{x^2-2x-3}\,dx$.

Here we note that $x^2 - 2x - 3 = (x+1)(x-3)$, so we let

$$\frac{-4}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}, \tag{1}$$

where A and B are constants we have to find. Multiplying both sides of (1) by $x^2 - 2x - 3$ we obtain

$$-4 = A(x-3) + B(x+1). \tag{2}$$

There are now two ways we can proceed. There is a quick way that will work in lots of cases and there is a slightly longer method that will work in all cases.

We will first look at the quick method.
If we let $x = 3$ in (2), we obtain $-4 = B(3 + 1)$, so $B = -1$.
Then if we let $x = -1$ in (2), we obtain $-4 = A(-1 - 3)$, so $A = 1$.
Thus we have $\dfrac{-4}{x^2 - 2x - 3} = \dfrac{1}{x + 1} + \dfrac{-1}{x - 3}$.
Before we go ahead and perform the integration, we will look at the other method (which will work in all cases no matter how complicated). What we do is to rewrite (2) as

$$-4 = (A + B)x + (-3A + B). \tag{3}$$

That is we collect the terms in x together and the constant terms together. If we now look at the coefficient of x on either side of (3) we see that $0 = A + B$ (the term on the left is $0x$) and if we look at the constant term on either side of (3), we see that $-4 = -3A + B$. Thus we obtain the simultaneous equations $0 = A + B$ and $-4 = -3A + B$. Solving these we get $A = 1$ and $B = -1$ as above.
We will now finish the job and perform the integration.

$$\int \frac{-4}{x^2 - 2x - 3}\, dx = \int \frac{1}{x + 1}\, dx + \int \frac{-1}{x - 3}\, dx = \ln(x + 1) - \ln(x - 3) + c.$$

I performed the integrations 'by inspection', but if you can't spot them, you can use the substitutions $u = x + 1$ and $u = x - 3$, respectively.

Next let us look at an example where $p(x)$ has three linear factors. Note that I won't expect you to factor a cubic - in cases like this I will always give you at least one of the factors. I will also keep all the examples in this section as indefinite integration examples, so we can focus on the partial fractions and not have to worry about substituting in the upper and lower limits. There is nothing difficult about this however, once you have done the integration, it is not a problem.

Example 4.7.3. Find the indefinite integral $\displaystyle\int \frac{3x^2 + 12x + 11}{(x + 1)(x + 2)(x + 3)}\, dx$.
First note that the numerator is not a multiple of the derivative of the denominator, so integration by substitution will not work and we have to use partial fractions. In this case we let

$$\frac{3x^2 + 12x + 11}{(x + 1)(x + 2)(x + 3)} = \frac{A}{x + 1} + \frac{B}{x + 2} + \frac{C}{x + 3}, \tag{4}$$

where A, B and C are constants we have to find. Multiplying both sides of (4) by $(x + 1)(x + 2)(x + 3)$ we obtain

$$3x^2 + 12x + 11 = A(x + 2)(x + 3) + B(x + 1)(x + 3) + C(x + 1)(x + 2). \tag{5}$$

The quick method will work again in this case, so let us do it this way first.
If we let $x = -1$ in (5), we obtain $3 - 12 + 11 = A(-1 + 2)(-1 + 3))$, so $A = 1$.
Then if we let $x = -2$ in (5), we obtain $12 - 24 + 11 = B(-2 + 1)(-2 + 3)$, so $B = 1$.
Finally if we let $x = -3$ in (5), we obtain $27 - 36 + 11 = C(-3 + 1)(-3 + 2)$, so $C = 1$.

Thus we have $\dfrac{3x^2+12x+11}{(x+1)(x+2)(x+3)} = \dfrac{1}{x+1} + \dfrac{1}{x+2} + \dfrac{1}{x+3}$.
The other method is as follows. If we expand the right hand side of (5) and collect together terms in x^2, x and the constant terms we obtain

$$(6) \qquad 3x^2 + 12x + 11 = (A+B+C)x^2 + (5A+4B+3C)x + (6A+3B+2C).$$

Then comparing terms in x^2, x and the constant terms in (6), we obtain the simultaneous equations $A+B+C=3$, $5A+4B+3C=12$ and $6A+3B+2C=11$. These can be solved to obtain $A=B=C=1$ as before.
After using either of these methods, we can perform the integration:

$$\begin{aligned}\int \frac{3x^2+12x+11}{(x+1)(x+2)(x+3)}\,dx &= \int \frac{1}{x+1}\,dx + \int \frac{1}{x+2}\,dx + \int \frac{1}{x+3}\,dx \\ &= \ln(x+1) + \ln(x+2) + \ln(x+3) + c.\end{aligned}$$

Next we will look at the case where $g(x)$ has a repeated factor.

Example 4.7.4. Find the indefinite integral $\displaystyle\int \frac{x-2}{(x-1)^2}\,dx$.
Again note that the numerator is not a multiple of the derivative of the denominator, so integration by substitution will not work and we have to use partial fractions. When we have a repeated factor in the denominator, we let

$$(7) \qquad \frac{x-2}{(x-1)^2} = \frac{A}{(x-1)^2} + \frac{B}{(x-1)}.$$

Note that a similar method is used for higher powers, we just need three terms for a third power and so on. Multiplying both sides of (4) by $(x-1)^2$ we obtain

$$(8) \qquad x - 2 = A + B(x-1).$$

Unfortunately when we have a repeated root then the simple method of finding A and B will not work completely. We can let $x=1$ in (8) to obtain $1-2=A$, so that $A=1$ but then we still need to use the other method of comparing coefficients to find B. We could use a hybrid method but perhaps it is more straightforward to go straight for the comparing coefficients method. Rearranging (8) we obtain

$$x - 2 = Bx + (A - B).$$

Then we obtain the simultaneous equations $B=1$ and $-2=A-B$. These have solution $A=-1$ and $B=1$, so $\dfrac{x-2}{(x-1)^2} = \dfrac{-1}{(x-1)^2} + \dfrac{1}{(x-1)}$. We can now perform the integration.

$$\begin{aligned}\int \frac{x-2}{(x-1)^2}\,dx &= \int \frac{-1}{(x-1)^2}\,dx + \int \frac{1}{(x-1)}\,dx \\ &= \frac{1}{x-1} + \ln(x-1) + c.\end{aligned}$$

I performed the last two integrations ‘by inspection’, but you can use the substitution $u = x-1$ if you can’t spot them.

For our final example, we will look at the case where $g(x)$ has a quadratic factor that can't be factorised.

Example 4.7.5. Find the indefinite integral $\displaystyle\int \frac{3x^2-4x+2}{(x^2-x+1)(x-1)}\,dx$.
In this case we let

$$\frac{3x^2-4x+2}{(x^2-x+1)(x-1)} = \frac{Ax+B}{x^2-x+1} + \frac{C}{x-1}. \tag{9}$$

Again the simple method for finding A, B and C will not work here, so we have to use the method of comparing coefficients. Multiplying both sides of (9) by $(x^2-x+1)(x-1)$ we obtain

$$3x^2-4x+2 = (Ax+B)(x-1) + C(x^2-x+1).$$

Multiplying this out and collecting terms we get

$$3x^2-4x+2 = (A+C)x^2 + (-A+B-C)x + (-B+C).$$

This yields the simultaneous equations $3 = A+C$, $-4 = -A+B-C$ and $2 = -B+C$. These can be solved to give $A=2$, $B=-1$ and $C=1$, so we have $\dfrac{3x^2-4x+2}{(x^2-x+1)(x-1)} = \dfrac{2x-1}{x^2-x+1} + \dfrac{1}{x-1}$. We can now perform the integration to get

$$\begin{aligned}\int \frac{3x^2-4x+2}{(x^2-x+1)(x-1)}\,dx &= \int \frac{2x-1}{x^2-x+1}\,dx + \int \frac{1}{x-1}\,dx \\ &= \ln(x^2-x+1) + \ln(x-1) + c.\end{aligned}$$

Note you can use the substitution $u = x^2-x+1$ for the second last integral if you can't spot it.

4.8. **Finding Areas and Volumes.**

In the introduction to this chapter, we noted that integration is essentially a method of calculating the area between the graph of a function and the x-axis. However we also showed that if we want to calculate areas then we have to be a little careful since integration finds 'signed areas'. That is if the function lies below the x-axis then the integral is negative. We will now look at a couple of examples where we want to calculate actual areas rather than signed areas.

Example 4.8.1. Find the area lying between the graph of $f(x) = x^3$ and the x-axis between the points $x=-1$ and $x=1$.
We first note that if $x<0$ then $x^3<0$ and if $x>0$ then $x^3>0$. So the graph of $f(x)$ lies below the x-axis for $x \in [-1,0)$ and above the x-axis for $x \in (0,1]$. We have to take account of the fact that it lies below the x-axis for $x \in [-1,0)$ by taking minus the integral for that range of x (since we are looking for the actual area rather than the signed area).

Hence the actual area is

$$\begin{aligned}
-\int_{-1}^{0} x^3\,dx + \int_{0}^{1} x^3\,dx &= -\left[\frac{x^4}{4}\right]_{-1}^{0} + \left[\frac{x^4}{4}\right]_{0}^{1} \\
&= -\left(0 - \frac{(-1)^4}{4}\right) + \left(\frac{1^4}{4} - 0\right) \\
&= \frac{1}{4} + \frac{1}{4} \\
&= \frac{1}{2}.
\end{aligned}$$

Warning 4.8.2. If we calculating an actual area (rather than a signed area), then we must get a positive answer. If we don't then we must have gone wrong somewhere.

Here is another example.

Example 4.8.3. Find the area lying between the graph of $f(x) = \sin(2x)$ and the x-axis between the points $x = -\frac{\pi}{2}$ and $x = 0$.
In this case the graph of f lies below the x-axis throughout the region of interest. Hence we just take minus the integral. So the area is

$$\begin{aligned}
-\int_{-\frac{\pi}{2}}^{0} \sin(2x)\,dx &= -\left[-\frac{1}{2}\cos(2x)\right]_{-\frac{\pi}{2}}^{0} \\
&= -\left[-\frac{1}{2}\cos(0) - \left(-\frac{1}{2}\cos(-\pi)\right)\right] \\
&= -\left[-\frac{1}{2} - \left(\frac{1}{2}\right)\right] \\
&= 1.
\end{aligned}$$

Here is one final example.

Example 4.8.4. Find the area lying between the graph of $f(x) = x^3 + x^2 - 9x - 9$ and the x-axis between the points $x = -2$ and $x = 2$ given that the graph of this function only crosses the x-axis at $x = -1$ in the interval $[-2, 2]$.
Here we are given that the graph of f only cuts the x-axis once in the region of interest. Thus it either lies above or below the x-axis between -2 and -1 and the opposite between -1 and 2. To decide which, we just have to check another point. In this case zero is easiest to check. Since $f(0) = -9 < 0$, it follows that f lies below the x-axis in the interval $(-1, 2]$ and above the x-axis in the interval $[-2, -1)$. Thus

the area is

$$
\begin{aligned}
&\int_{-2}^{-1} x^3+x^2-9x-9\,dx \ -\int_{-1}^{2} x^3+x^2-9x-9\,dx \\
&=\left[\frac{1}{4}x^4+\frac{1}{3}x^3-\frac{9}{2}x^2-9x\right]_{-2}^{-1}-\left[\frac{1}{4}x^4+\frac{1}{3}x^3-\frac{9}{2}x^2-9x\right]_{-1}^{2} \\
&=\left[\left(\frac{1}{4}(-1)^4+\frac{1}{3}(-1)^3-\frac{9}{2}(-1)^2-9(-1)\right)\right. \\
&\quad\left.-\left(\frac{1}{4}(-2)^4+\frac{1}{3}(-2)^3-\frac{9}{2}(-2)^2-9(-2)\right)\right] \\
&\quad-\left[\left(\frac{1}{4}(2)^4+\frac{1}{3}(2)^3-\frac{9}{2}(2)^2-9(2)\right)\right. \\
&\quad\quad\left.-\left(\frac{1}{4}(-1)^4+\frac{1}{3}(-1)^3-\frac{9}{2}(-1)^2-9(-1)\right)\right] \\
&=\left[\frac{53}{12}-\frac{4}{3}\right]-\left[-\frac{88}{3}-\frac{53}{12}\right] \\
&=\frac{221}{6}.
\end{aligned}
$$

We can also use definite integration to find volumes. What we will look at in this section is finding the volumes created when graphs are rotated about the x-axis but there are many other techniques available.

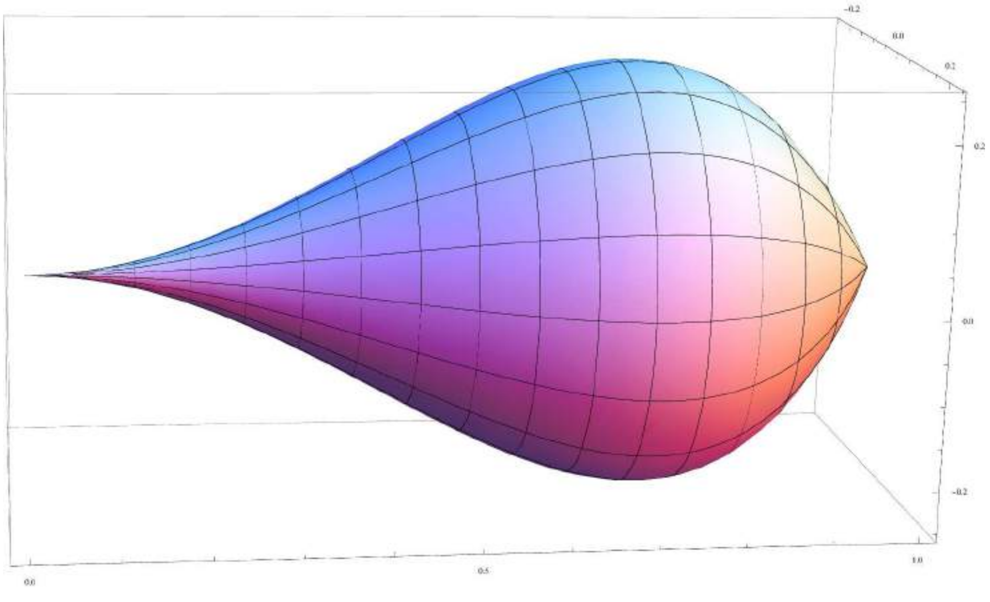

FIGURE 4. The volume of revolution of the function $f(x) = x^4 - x^2$ between $x = 0$ and $x = 1$.

Figure 4 shows the graph of the the function $f(x) = x^4 - x^2$ rotated about the x-axis between the points $x = 0$ and $x = 1$. Luckily there is a simple formula that allows us to find the volume of the body obtained when any function is rotated about the x-axis.

Theorem 4.8.5 (Volume of Solid of Revolution)**.** *The volume V obtained when a function f is rotated about the x-axis between the points $x = a$ and $x = b$ is given by*

$$V = \pi \int_a^b f(x)^2\, dx.$$

Remark 4.8.6.
1. Since the function is squared, we don't have to worry about whether the function is positive or negative, as we did when calculating areas.
2. Since the area of a circle of radius $f(x)$ is $\pi f(x)^2$, this formula is really just obtained by summing up the volumes of a series of disks perpendicular to the x-axis, in a similar fashion to the way the area under a curve was obtained by summing up the areas of a series of rectangles.

Let us now show how this works in practice by calculating the volume of the solid shown in Figure 4

Example 4.8.7. Find the volume of revolution of the function $f(x) = x^4 - x^2$ about the x-axis between $x = 0$ and $x = 1$.
In this case we use Theorem 4.8.5 with $a = 0$, $b = 1$ and $f(x) = x^4 - x^2$. Hence the volume is

$$\begin{aligned}
V &= \pi \int_0^1 \left(x^4 - x^2\right)^2\, dx \\
&= \pi \int_0^1 x^8 - 2x^6 + x^4\, dx \\
&= \pi \left[\frac{1}{9}x^9 - \frac{2}{7}x^7 + \frac{1}{5}x^5\right]_0^1 \\
&= \pi \left[\left(\frac{1}{9} - \frac{2}{7} + \frac{1}{5}\right) - (0 - 0 + 0)\right] \\
&= \frac{8\pi}{315}.
\end{aligned}$$

Warning 4.8.8. As was the case when calculating actual areas, we must get a positive answer when calculating volumes. If we don't then we have gone wrong somewhere.

Let us do one more example.

Example 4.8.9. Find the volume of revolution of the function $f(x) = 2e^{2x}$ about the x-axis between $x = 1$ and $x = 3$.
In this case we use Theorem 4.8.5 with $a = 1$, $b = 3$ and $f(x) = e^{2x}$. Hence the

volume is

$$V = \pi \int_1^3 \left(2e^{2x}\right)^2 \, dx = \pi \int_1^3 4e^{4x} \, dx = \pi \left[e^{4x}\right]_1^3 = \pi \left(e^{12} - e^4\right).$$